Motion of Polygonal Prisms Down Inclined Planes

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EDITOR'S NOTE:

The following is a project nominated for publication by the administrators behind Science One - a first-year science program at the University of British Columbia. Science One is an integrated program, where students have a chance to complete a research project in their second term.

Abstract

An experiment was carried out to determine the rolling times of square, hexagonal and circular prisms down inclined ramps of various angles. Each roll was recorded on video and rolling times were determined by counting frames. A model was created to predict the times of descent for n -sided prisms under the assumption of persistent contact between prism and ramp. A second model was created without this assumption. Animations of the prisms' motion were created according to the models. Predicted times of the models approached experimental times for polygonal prisms as the number of sides increased. Notable properties of the rolling motions are discussed, and reasons for discrepancies between models and results are addressed.

Key Words

Polygonal Prisms, Inclined Plane, Restitution Coefficient, Modeling

Introduction

Rotational and collisional interactions underlie the mechanics of systems that span a vast range of scales, from the rolling of pencils across a desk to the falling of boulders in a landslide. A rolling prism demonstrates inelastic collisions between its faces and the ramp surface. Between collisions, the prism touches the ramp along a single edge and rotates due to gravity. At greater angular speeds, the prism loses contact with the ramp. The prism is expected to reach a terminal velocity since a proportion of kinetic energy is lost in each collision [1]. An initial "walking model" was developed, which assumes contact between the prism and the ramp at all times. A more sophisticated "running model" was developed since the prism was observed to lose contact with the ramp in the experiment. Rolling times of the prisms were measured and were shown to depend on properties of the prisms and the angle of the ramp.

Materials and Methods

The prisms and the ramp consist of solid wood. The prisms'

masses were determined using a top-loading balance and dimensions were measured using a caliper. The average radius $\bar{r}$ is the mean distance between the center and corners of each face, as shown in Table 1.

Table 1: Prisms' mass and dimensions, in which $u[x]$ denotes the uncertainty of x .

	mass (kg)	$u[\text{mass}]$ (kg)	$\bar{r}$ (m)	$u[\bar{r}]$ (m)
Square Prism	0.02520	0.00003	0.01733	0.00005
Hexagonal Prism	0.02800	0.00003	0.01413	0.00006
Circular Cylinder	0.02330	0.00003	0.0119	0.0003

Sandpaper of 180 grit was glued to the ramp to prevent sliding. The length of the ramp was measured to be 0.2790 m. The inclination angle θ was determined trigonometrically by adjusting the vertical distance between the ground and a reference point on the ramp. The static friction coefficient μ_f between the sandpaper and the prisms was determined to be 0.923 by finding the maximum inclination angle such that the prism remains stationary and does not slide. Prisms were rolled with one corner touching the top edge of the sandpaper and released at a starting angle φ_{0i} , defined as the angle between the vertical line through the contact corner and the radius segment of the corner, as shown in Figure 1. If the starting position is close to equilibrium, small variations in φ_{0i} can cause large differences in rolling times. Starting angles of around 5° were marked on the prisms and exact starting angles were determined trigonometrically. The exact angles are presented in Table 2.

Figure 1: Illustration of φ_{0i} and r .

Table 2: The initial starting angle φ_{0i} and its uncertainty $u[\varphi_{0i}]$ at which the prisms are released.

	φ_{0i} (rad)	$u[\varphi_{0i}]$ (rad)
Square Prism	0.08085	0.00001
Hexagonal Prism	0.11350	0.00003
Circular Cylinder	θ	-

Each roll was filmed and frames were counted using QuickTime to determine the time between the prism's release and the instant when the prism's 3D center of mass is directly above the lower edge of the sandpaper. Frame numbers were converted to seconds using the phone's recording rate of 240.0 frames/second. Trials were discarded if the prism rolled sideways off the sandpaper as a result of imperfect release. The proportion of energy lost in face-on collisions is accounted by the restitution coefficient e_F , which is defined as the ratio of the angular speed right after a collision over the angular speed right before the collision. It is approximated by carefully releasing the prism near equilibrium and observing whether the prism reaches its next equilibrium after one collision. The equilibrium is defined as the position at which the prism balances on its edge. The ramp is adjusted until a critical inclination angle θ_c is reached where upon release, the prism just reaches its next equilibrium. For an n -sided polygonal prism,

$$e_F = \frac{\sqrt{1 - \cos\left(\theta_c - \frac{\pi}{n}\right)}}{\sqrt{1 - \cos\left(\theta_c + \frac{\pi}{n}\right)}} \quad (1)$$

Since square and hexagonal prisms are not perfectly shaped, e_F was measured for each face of the prism and an average was obtained as $\bar{e}_F$. When a "running" prism regains contact with the ramp, its energy loss is described by the edge-on coefficient e_E , which has not been determined in the experiment and is assumed to have the same value as $\bar{e}_F$ in the models.

Models

The motion of a regular n -sided polygonal prism down an inclined plane is simplified by considering the movement of a two-dimensional polygon down a linear ramp. Given a release angle φ_i and an angular speed ω_i , the walking model predicts the angle φ_{i+1} after a small time interval Δt , by assuming a constant angular speed over this interval. The angular speed ω_{i+1} is derived from energy conservation, giving

$$\omega_{i+1} = \sqrt{\omega_i^2 + \frac{2Mgr}{I} (\cos(\varphi_i) - \cos(\varphi_{i+1}))}, \quad (2)$$

where M is the mass of the prism and I is its moment of inertia about an edge, given by

$$I = \frac{Mr^2}{6} (2\cos^2\left(\frac{\pi}{n}\right) + 7) \quad [2]. \quad (3)$$

With initial φ_0 and ω_0 , prognosticated values for φ_i and ω_i are determined using Euler integration. When $\varphi = \theta + \frac{\pi}{n}$, a face-on collision occurs and φ is reset to $\theta - \frac{\pi}{n}$ as the corner of contact switches, where n is the number of sides of the prism. To account for energy loss, the angular speed right after a collision is given by multiplying the angular speed immediately before the collision with e_F . The horizontal displacement of the center of mass, Δx , is tracked until the prism reaches the end of the ramp. The cylinder model uses $n = 1000$ and assumes $e_F = 1$ since negligible energy should be lost during the rolling motion of the cylinder. Approximating a cylinder with $n = 1000$ is sufficient since the results agreed to 4 significant figures when $n = 10000$ is used.

The running model, unlike the walking model, accounts for prisms' loss of contact with the ramp. When $\omega > \sqrt{\frac{g\cos(\varphi)}{r}}$, the prism loses contact since the centripetal acceleration about the corner of rotation overcomes the component of gravity along the effective radius [3]. At the instant of losing contact, the horizontal and vertical speeds of the center of mass are respectively $v_{x0} = \omega_0 r \cos(\varphi_0)$ and $v_{y0} = -\omega_0 r \sin(\varphi_0)$, where ω_0 is the angular speed and φ_0 is the angle at the moment of losing contact. The height of each corner above the ramp as a function of time $H_m(t)$ is described by

$$H_m(t) = r\cos(\varphi_0) + r\sin(\varphi_0)\tan(\theta) + v_{y0}t - \frac{1}{2}gt^2 + \tan(\theta)v_{x0}t + r\cos(\mu_m)\tan(\theta) - r\sin(\mu_m), \quad (4)$$

where t is the elapsed time since losing contact, m is the index of each corner, and μ_m is the clockwise angular distance from the horizontal. The last corner in contact with the ramp is indexed 0 and subsequent corners are indexed counter-clockwise as shown in Figure 2.

Figure 2: Illustration of H and μ .

Figure 3: Comparison of model and experiment times.

(A) Rolling times for the square prism at various ramp angles.

(B) Rolling times for the hexagonal prism at various ramp angles.

(C) Rolling times for the cylinder at various ramp angles.

Model-predicted rolling times are determined by inputting the corresponding θ , l , $\bar{e}_F$ for each roll. Each experimental data point is the average of four or five trials. Error bars of experiment represent standard mean error; error bars for model represent 68% confidence intervals.

The angular distance for each corner is given by

$$\mu_m = \frac{\pi}{2} + \varphi_0 - m \frac{2\pi}{n} + \omega_0 t. \quad (5)$$

$H_m(t)$ is determined by updating μ_m under the assumption of constant angular speed through the air. The horizontal displacement of the center of mass is updated accordingly until any H_m reaches zero and the prism regains contact. Immediately, the new initial walking angle φ_{0r} and angular speed ω_{0r} are

$$\varphi_{0r} = -m \frac{2\pi}{n} + \varphi_0 + \omega_0 t_f, \quad (6)$$

$$\omega_{0r} = \sqrt{\frac{2}{I} e_E^2 \left(\frac{1}{2} M v_{x0}^2 + \frac{1}{2} M (v_{y0} - g t_f)^2 + \frac{1}{2} I_c \omega_0^2 \right)}. \quad (7)$$

where t_f is the total flying time, and I_c is the moment of inertia about the central axis of the prism, obtained by reverse application of the Parallel Axis Theorem to Equation (3). If ω_{0r} is greater than the critical walking speed, the running model starts again; otherwise, the prism continues with the walking model until the next moment of losing contact.

Results and Discussion

Rolling times for models and experiments are shown in Figure 3, and the average restitution coefficients determined for square and hexagonal prisms are shown in Table 3.

Table 3: $\overline{e_F}$ for each prism.

	$\overline{e_F}$	$u[\overline{e_F}]$
Square prism	0.656	0.030
Hexagonal prism	0.856	0.026

The uncertainties in radius, mass, rolling length, starting angle, and $\overline{e_F}$ all contribute to errors in modelling; however, the relative uncertainty in $\overline{e_F}$ is an order of magnitude greater than the uncertainties in the other values. Hence, the error in model times are determined by inputting the upper and lower bounds of $\overline{e_F}$, and taking other uncertainties as negligible. Error bars are excluded from the cylinder model since it is assumed that $e_F = 1$.

Root mean square deviation (RMSD) measures the differences between model and experiment. Running models have lower RMSD than walking models and are therefore more accurate, as shown in Table 4. RMSD decreases when n increases, which suggests that both models may better predict rolling times for larger n .

Table 4: RMSD.

	Square prism	Hexagonal prism	Circular prism
Walking model	0.1914	0.1559	0.018580
Running model	0.1489	0.1295	0.018577

In general, the running model agrees with the walking model for small θ , but predicts rolling times better than the walking models for

larger θ . The overall trend of the cylinder model agrees with the experiment; however, most of the predicted values lie outside the experiment's error bars. This may be due to the wood cylinder's non-uniform density, which could result in a smaller moment of inertia. The running model is more effective for smaller n , yet both models make more accurate predictions as n increases.

The underestimation of rolling times at smaller θ may result from double bouncing. A double bounce consists of a face-on collision followed immediately by an edge-on collision, as shown in Figure 4. Both models only capture the dynamics of single bounces; however, additional energy is lost in double bouncing. This lower energy profile translates to greater rolling times in experiments. In addition, the walking model overestimates rolling times for large θ since it does not account for the running motion that prisms exhibit in experiments. The overestimation in the running model for large θ may be due to equating e_E with e_F in the model. An edge-on collision typically experiences a smaller energy loss, as shown by the maximum rebound heights in Figure 5. As such, e_E must be greater than e_F . Furthermore, e_E is difficult to be experimentally quantified. Edge-on collisions do not always occur perpendicularly, so energy loss and e_E may vary with collision angle.

Figure 4: Snapshots of double bouncing with $\theta \approx 23^\circ$, in chronological order from left to right.

Figure 5: Maximum rebound for a square prism in a face-on collision (left) and edge-on collision (right) dropped from the same height.

Since the restitution coefficient is the largest source of error in the experiment, predicted rolling times for the walking model were plotted against varying e_F to determine possible relationships. All other parameters were kept the same as in the walking model for the square prism. Rolling time decreases with greater restitution coefficient, as shown in Figure 6. For values of e_F below 0.30, the prism is unable to reach its next equilibrium position. A restitution coefficient

cient closer to one results in a more constant curve at the experimental angles, shown in Figure 7.

Figure 6: Model-predicted rolling times of square prism for various restitution coefficients $0.31 \leq e_F \leq 0.99$, with $l = 0.279\text{m}$, $\theta = 25^\circ$.

Figure 7: Model-predicted rolling times for the square prism at various ramp angles with $e_F = 0.4, 0.65, 0.8$; $l = 0.279\text{m}$.

For $e_F < 1$, the prism approaches a terminal velocity in the limit of large t . For the walking model, the kinetic energy E_k of the prism after each collision plateaus to

$$E_{k\infty} = 2 \frac{e_F^2}{1-e_F^2} mgr \cdot \sin\left(\frac{\pi}{n}\right) \cdot \sin(\theta). \quad [1] \quad (8)$$

The terminal angular speed $\omega_{\varphi\infty}$ for a specific position φ is given by

$$\omega_{\varphi\infty}^2 = \frac{2}{I} (E_{k\infty} + mgr \left(\cos\left(\theta - \frac{\pi}{n}\right) - \cos(\varphi) \right)). \quad (9)$$

The dependence of the terminal velocity on e_F is shown in Figure 8. The plot is not perfectly linear since $\omega_{\varphi\infty}$ cannot be precisely reached with a finite ramp. The slope of the plot is reasonably close to $\frac{4}{I} mgr \cdot \sin\left(\frac{\pi}{4}\right) \cdot \sin(\theta)$, which is in agreement with calculations.

Figure 8: Relationship between terminal velocity and restitution coefficient for the square prism walking model, $0.31 \leq e_F \leq 0.99$, $l = 2.79\text{m}$, $\theta = 25^\circ$.

Conclusion

The rolling times of square, hexagonal and circular prisms at various inclination angles follow a trend similar to that obtained from the walking and running models. The cylinder had the strongest agreement with the models. The running model provides better predictions for the rolling times of square and hexagonal prisms, but the two models converge with sufficiently large n . Further investigation into e_F and quantification of double-bounces will lead to more effective models.

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References

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REVIEW

by Dr. Robert Raussendorf

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The authors derive two mathematical models for n -sided prisms rolling down an inclined plane, and test their predictions of roll times against experiment. The video footage from the experiments also allows studying individual collisions between prisms and plane in detail.

The agreement between theory and experiment is good, in particular for the significantly more involved “running model”, but it is not perfect. For 4 and 6-sided prisms, the predicted roll times vary more weakly with the incline angle than the experimental data shows.

A small suggestion (optional) therefore is the following: The authors point out in Fig. 7 that lower restitution coefficients lead to steeper curves, and the question thus arises if better agreement between theory and experiment could be obtained by fitting the parameters θ_e , θ_f . Perhaps it is worth trying. A physical justification would be that the restitution coefficients could be velocity-dependent. The coefficient θ_f is determined in a near-static experiment whereas the θ_e that enters the simulation is at higher velocities.

I like the aspect that video footage is included in the figures, to illustrate the double-bounce.

Overall, this is substantial work, of high quality. The models for the rolling process are worked out in detail, and the resulting differential equations are integrated numerically. The predictions are compared with experiment, and match to a good degree of accuracy. The remaining discrepancies are analyzed, and possible refinements of the model are suggested to account for them. Therefore, I recommend this manuscript for publication in the STEM Fellowship Journal.

SUPPLEMENTAL DERIVATIONS

Derivation of Walking Model

The instantaneous angular velocity is approximated by considering the mechanical energy of the polygon. At some instant t_i , the polygon has angle φ_i and angular speed ω_i , and has mechanical energy

$$U_i = \frac{1}{2} I \omega_i^2 + mgr \cos \varphi_i + C, \quad (\text{i})$$

where I is the polygon's second area moment and C is the gravitational potential energy that the prism has when its center of mass is vertically aligned with the corner of contact. At a time Δt after this instant, the subsequent angle φ_{i+1} can be approximated by Euler integration as

$$\varphi_{i+1} = \varphi_i + \omega_i \Delta t. \quad (\text{ii})$$

This can be used to approximate the energy of the polygon at t_{i+1} giving

$$U_{i+1} = \frac{1}{2} I \omega_{i+1}^2 + mgr \cos (\varphi_i + \omega_i \Delta t) + C. \quad (\text{iii})$$

Neglecting air drag, mechanical energy is conserved within one roll; this model assumes that sliding never occurs, so friction does not do work. As a result, $U_i = U_{i+1}$ and the subsequent angular speed can be approximated by

$$\omega_{i+1} = \sqrt{\omega_i^2 + \frac{2mgr}{I} (\cos \varphi_i - \cos (\varphi_i + \omega_i \Delta t))}. \quad (\text{iv})$$

Eq (ii) and (iv) are used to calculate φ and ω for subsequent instances.

Derivation of Restitution Coefficient (Face-on Collision):

Assuming that the experiment is ideal, a prism balanced at its equilibrium on a ramp inclined to θ would experience an infinitesimally small impulse and stop precisely at its next equilibrium. Before the collision, potential energy is fully converted to kinetic energy and

$$mg(1 - \cos(\theta + \frac{\pi}{n})) = \frac{1}{2} I \omega_i^2, \quad (\text{v})$$

where n is the number of sides on the polygon and ω_i is the angular speed right before a collision. After the collision, kinetic energy is fully converted to potential energy, yielding

$$\frac{1}{2} I \omega_f^2 = mg(1 - \cos(\theta - \frac{\pi}{n})). \quad (\text{vi})$$

Given $e_F \equiv \frac{\omega_f}{\omega_i}$,

$$e_F = \frac{\sqrt{1 - \cos(\theta - \frac{\pi}{n})}}{\sqrt{1 - \cos(\theta + \frac{\pi}{n})}}. \quad (\text{vii})$$

Derivation of Running Model

Criterion of Losing Contact:

Figure I.

Centripetal acceleration: $a_c = \omega^2 r$;

Gravitational component: $g_r = g \cos \varphi$.

When $a_c > g_r$, $\omega > \sqrt{\frac{g \cos \varphi}{r}}$ (in the experiment, $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$, and $\cos \varphi > 0$).

Initial Velocities

Figure II.

φ in Figure II represents φ_0 .

$\alpha = \varphi_0$.

$v_{x0} = v \cos \varphi_0$, $v_{y0} = -v \sin \varphi_0$.

Given $v = \omega r$, $v_{x0} = \omega r \cos \varphi_0$ and $v_{y0} = -\omega r \sin \varphi_0$.

Derivation of $H(t)$:

H was broken down into components as shown in Figure III.

Figure III.

$$H = h - L + K.$$

h_0 , L_0 and K_0 :

Figure IV.

$$L_0 = r \sin(\mu_0).$$

$$K_0 = \overline{Ek_1} \tan(\theta) = \overline{GF} \tan(\theta) = r \cos(\mu_0) \tan(\theta).$$

$$h_0 = \overline{AJ} + \overline{IE} = r \cos(\varphi_0) + \overline{AI} \tan(\theta) = r \cos(\varphi_0) + \overline{JG} \tan(\theta) = r \cos(\varphi_0) + r \sin(\varphi_0) \tan(\theta).$$

$$H_0 = h_0 - L_0 + K_0,$$

$$H_0 = r \cos(\varphi_0) + r \sin(\varphi_0) \tan(\theta) + r \cos(\mu_0) \tan(\theta) - r \sin(\mu_0), \quad (\text{viii})$$

where $\mu_0 = \varphi_0$ (for the square, but more generally,

$$\mu_0 = \pi - \left(\frac{\pi}{2} - \varphi_0 \right) - \frac{2\pi}{n} = \frac{\pi}{2} - \frac{2\pi}{n} + \varphi_0 \quad (\text{ix})$$

for any n -sided polygon).

$h(t)$, $L(t)$ and $K(t)$:

Let t be the time spent in the air and let $t = 0$ be the time instant of losing contact.

Figure V.

Neglecting air resistance, ω_0 and v_{x0} are constant while the polygon is in the air. As such,

$$\mu(t) = \mu_0 + \omega_0 t,$$

$$L(t) = r \sin(\mu),$$

$$K(t) = r \cos(\mu) \tan(\theta).$$

When the square moves forward, $h(t)$ increases. the horizontal displacement of the square results in an increase in the vertical distance between point G and E, which is equal to $v_{x0} t \tan(\theta)$. In addition, the vertical displacement of an object with initial vertical velocity v_{y0} is given by $v_{y0} t - \frac{1}{2} g t^2$. Combining the above,

$$h(t) = h_0 + v_{x0} t \tan(\theta) + v_{y0} t - \frac{1}{2} g t^2.$$

Overall,

$$H(t) = r \cos(\varphi_0) + r \sin(\varphi_0) \tan(\theta) + v_{x0} t \tan(\theta) + v_{y0} t - \frac{1}{2} g t^2 + r \cos(\mu) \tan(\theta) - r \sin(\mu), \quad (\text{x}).$$

Computationally, $H(t)$ starts at H_0 , and t increases by dt , using Euler integration until $H(t) = 0$. In essence, the program solves for t when $H(t) = 0$, and defines the first solution to be t_f . When the prism lands, it loses some kinetic energy modelled by the coefficient e_E . Immediately before the collision, the prism possesses kinetic energy E_k . Immediately after the collision, the prism has kinetic energy $E_r = e_E^2 E_k$ (subscript r denotes “running”),

$$E_r = e_E^2 \left(\frac{1}{2} m v_{x0}^2 + \frac{1}{2} m (v_{y0} - g t_f)^2 + \frac{1}{2} I_c \omega_0^2 \right). \quad (\text{xi})$$

Immediately after the prism lands, the angular speed with respect to the corner of contact is

$\omega_{r0} = \sqrt{\frac{2}{I} E_r}$. The angular position is redefined as φ_{0r} when the prism lands. The two possible cases are shown in figures VI.1 and VI.2.

Figure VI.1, VI.2.

φ_0 was defined to be negative when the center of mass is behind the equilibrium, and positive otherwise. As such,

$$\varphi_{0r} = -\left(\frac{\pi}{2} - \mu_f\right), \quad (\text{xii})$$

$$\text{where } \mu_f = \mu_0 + \omega_0 t_f. \quad (\text{xiii})$$

Right after landing, there are two possible scenarios. If $\omega_{0r} > \sqrt{\frac{g \cos \varphi_{0r}}{r}}$, the prism would immediately lose contact again; if $\omega_{0r} \leq \sqrt{\frac{g \cos \varphi_{0r}}{r}}$, the prism will roll until it loses contact again. In the first scenario, the prism would be modelled by the running model. In the second scenario, the walking model would be applied until the prism runs again. In the first scenario, it is possible that φ_0 would start negative, as shown in Figure VII;

Figure VII.

however, all relevant equations still apply.

All the above equations hold true for any n -sided regular polygon. However, as discussed in paper, edge-skipping must be accounted for as follows.

Figure VIII.

Define μ to be negative when the corner is above the horizontal line through G. For example, when $m = 2$, $n = 6$,

$$\mu_{2,i} = \mu_{1,i} - \frac{\pi}{3} = \varphi_0 - \frac{\pi}{6}.$$

More generally,

$$\mu_m = \mu_{mi} + \omega_0 t = \frac{\pi}{2} + \varphi_0 - m \frac{2\pi}{n} + \omega_0 t \quad (\text{xiv})$$

Similarly, $H_m(t) = h - L + K$, that is,

$$H_m(t) = r \cos(\varphi_0) + r \sin(\varphi_0) \tan(\theta) + v_{y0} t - \frac{1}{2} g t^2 + v_{x0} t * \tan \theta - r \sin \mu_m + r \cos \mu_m \tan \theta. \quad (\text{xv})$$

The landing angle φ_{0r} can be redefined as

$$\varphi_{0r} = -\frac{\pi}{2} + \mu_{mi} + \omega_0 t_f \quad (\text{xvi})$$

where t_f is the time when any H_m reaches 0.

Derivation of Uncertainty of Starting Angle

The starting angle φ_0 is measured by drawing a right triangle with respect to φ_0 , and measuring the opposite-side length D and hypotenuse l of the triangle with a caliper to obtain

$$\varphi_0 = \sin^{-1}\left(\frac{D}{l}\right). \quad (\text{xx})$$

The uncertainty of $\varphi_0(D, l)$ is derived by manipulating the partial derivatives with respect to D and l ,

$$u[\varphi_0] = \sqrt{(u[D] \frac{d\varphi_0}{dD})^2 + (u[l] \frac{d\varphi_0}{dl})^2}. \quad (\text{xxi})$$

Derivation of Terminal Velocity

Let the kinetic energy gained before the first collision be E_0 , and the kinetic energy right after c^{th} collision be E_{kc} . Recall ΔE is the kinetic energy gained between each collision.

$$\begin{aligned} E_{k1} &= E_0 \cdot e_F^2, \\ E_{k2} &= (E_0 \cdot e_F^2 + \Delta E)e_F^2, \\ E_{k3} &= (E_0 \cdot e_F^2 + \Delta E \cdot e_F^2 + \Delta E)e_F^2, \\ E_{kc} &= E_0 \cdot e_F^{2c} + (\Delta E \cdot (e_F^2)^{c-1} + \Delta E \cdot (e_F^2)^{c-2} + \dots + \Delta E \cdot (e_F^2)). \end{aligned} \quad (\text{xxii})$$

$$\text{For } 0 < e_F < 1, E_{kc} = \Delta E \frac{e_F^2}{1-e_F^2} \text{ when } c \rightarrow \infty.$$

ΔE is equal to the potential energy lost between each collision.

Figure IX

As shown in Figure IX,

$$\Delta E = mgd,$$

$$d = a \sin \theta,$$

$$a = 2r \sin\left(\frac{\pi}{n}\right),$$

$$\Delta E = mg 2r \sin\left(\frac{\pi}{n}\right) \sin \theta. \quad (\text{xxiii})$$

The term $mgr(\cos(\theta - \frac{\pi}{n}) - \cos \varphi)$ in Eq (16) represents the potential energy difference between the position right after collision and the position with φ , as shown in Figure X, where $\varphi_0 = \theta - \frac{\pi}{n}$, and $D = r(\cos(\theta - \frac{\pi}{n}) - \cos \varphi)$.

Fig X

Derivation of Friction Coefficient

The friction coefficient is determined by slowly raising the inclined plane until the prism that is initially at rest starts to slide; the prism is placed with its longer edge parallel to the ramp as shown in Fig XI, and the angle θ at which the prism slides is recorded.

Let the friction coefficient be μ_f ,

$$F_f = \mu_f F_N,$$

$$F_N = g_y = F_g \cos \theta = mg \cos \theta.$$

At the instant before the prism slides, $F_f = g_x = mg \sin \theta$; therefore,

$$\mu_f mg \cos \theta = mg \sin \theta$$

$$\mu_f = \frac{\sin \theta}{\cos \theta} \quad (\text{xxiv})$$

Fig XI